THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Advanced Calculus I
Tutorial 10

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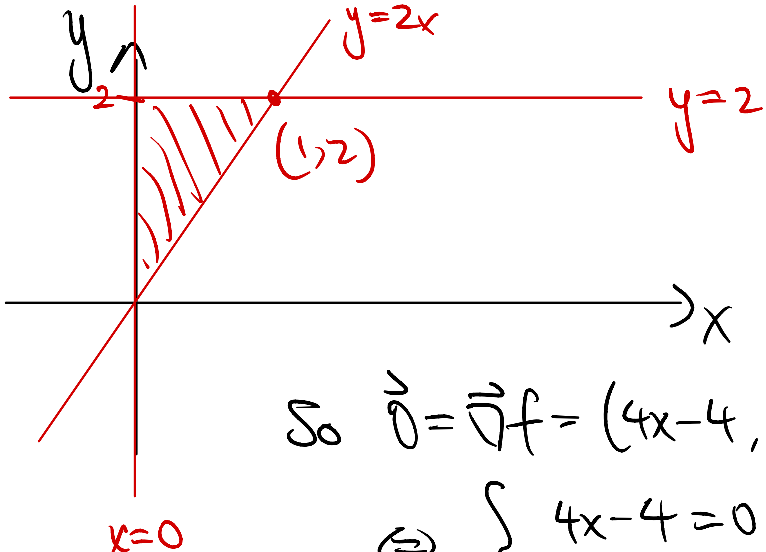
1. Find the absolute maxim and minima of the function $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0, y = 2, y = 2x$ in the first quadrant.

2. (a) Show that $(0,0)$ is a critical point of $f(x,y) = x^2 + kxy + y^2$ for any $k \in \mathbb{R}$.
   (b) For what values of k does the Second Derivative Test guarantee that f will have a saddle point at $(0,0)$, a local minimum at $(0,0)$?

3. Find the 2nd order Taylor polynomial near the origin of
   (a) $f(x,y) = e^x \ln(1+y)$;
   (b) $f(x,y) = \sin(x^2 + y^2)$.

4. Find the 2nd order Taylor polynomial of $f(x,y) = 4xy - x^4 - y^4$ near $(0,0)$ and also find and classify all of its critical points using the Second Derivative Test.

1. Find the absolute maxim and minima of the function $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0, y = 2, y = 2x$ in the first quadrant.



Critical Points:

f is a polynomial $\Rightarrow \vec{\nabla} f$ always exists.

$$\vec{\nabla} f = (4x-4,\ 2y-4).$$

So $\vec{0} = \vec{\nabla} f = (4x-4, 2y-4)$

$$\Leftrightarrow \begin{cases} 4x-4=0 \\ 2y-4=0 \end{cases} \Leftrightarrow \begin{cases} x=1 \\ y=2. \end{cases}$$

So $(1,2)$ is a critical point and

$$f(1,2) = 2\cdot 1^2 - 4\cdot 1 + 2^2 - 4\cdot 2 + 1 = -5.$$

Study f on bdry:

Along $x=0$:

$f(0,y) = y^2 - 4y + 1$ and $\left.\frac{df}{dy}\right|_{x=0} = 2y-4 = 0 \Leftrightarrow y=2.$

$f(0,2) = -3.$

Along $y=2$:

$f(x,2) = 2x^2 - 4x - 3$ and $\left.\frac{df}{dx}\right|_{y=2} = 4x-4 = 0 \Leftrightarrow x=1$

So $f(1,2) = -5.$

Along $y=2x$:

$f(x,2x) = 6x^2 - 12x + 1$ and $\left.\frac{df}{dx}\right|_{y=2x} = 12x - 12 = 0 \Leftrightarrow x=1.$ $f(1,2)$

So we conclude that f has absolute minimum at $(1,2)$ with

$f(1,2) = -3$.

and f has abs max at $(0,0)$ with $f(0,0) = 1$.

2. (a) Show that $(0,0)$ is a critical point of $f(x,y) = x^2 + kxy + y^2$ for any $k \in \mathbb{R}$.
   (b) For what values of k does the Second Derivative Test guarantee that f will have a saddle point at $(0,0)$, a local minimum at $(0,0)$?

a) $k=0$: $f(x,y) = x^2+y^2$ and $\vec{\nabla} f = (2x, 2y) = (0,0)$

$\Rightarrow (x,y) = (0,0)$.

So $(0,0)$ is a critical point.

$k \neq 0$: $f(x,y) = x^2 + kxy + y^2$ and $\vec{\nabla} f = (2x+ky, 2y+kx) = (0,0)$

$\Leftrightarrow \begin{cases} 2x+ky=0 & (1) \\ 2y+kx=0 & (2) \end{cases}$ $\quad (1) \Rightarrow x = \frac{-ky}{2}$

Sub into (2): $2y + k\left(\frac{-ky}{2}\right) = 0$

$\Leftrightarrow 2y - \frac{k^2 y}{2} = 0$

$\Leftrightarrow y\left(2 - \frac{k^2}{2}\right) = 0$

$\Rightarrow y=0$ or $k=4$.

If $y=0$, then $x=0$.

If $k=4$: then $\begin{cases} 2x+4y=0 & (1)' \\ 2y+4x=0 & (2)' \end{cases}$

and then $(1)' - 2(2)'$: $2x+4y-4y-8x=0$.

$\Rightarrow -6x=0 \Rightarrow x=0$.

$\Rightarrow y=0$.

So $(x,y)=(0,0)$ is a critical point.

b) Second Derivative Test: at critical point $\vec{a}$:

$f_{xx} < 0$ and $\det(Hf(\vec{a})) > 0 \Rightarrow \vec{a}$ local max.

$f_{xx} > 0$ and $\det(Hf(\vec{a})) > 0 \Rightarrow \vec{a}$ local min.

" " $\det(Hf(\vec{a})) < 0 \Rightarrow \vec{a}$ saddle pt.

" " $\det(Hf(\vec{a})) = 0 \Rightarrow$ inconclusive.

$$Hf(0,0) = \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} = \begin{bmatrix} 2 & k \\ k & 2 \end{bmatrix}$$

Note that $f_{xx} = 2 > 0$.

$\det(Hf(0,0)) = f_{xx}f_{yy} - f_{xy}f_{yx} = 2\cdot 2 - k\cdot k = 4 - k^2$.

$k < -2, k > 2$, $\det(Hf(0,0)) < 0 \Rightarrow$ saddle pt. $k = \pm 2$, inconclusive

$-2 < k < 2$, $\det(Hf(0,0)) > 0 \Rightarrow$ local min.

3. Find the 2nd order Taylor polynomial near the origin of

   (a) $f(x, y) = e^x \ln(1 + y)$;

   (b) $f(x, y) = \sin(x^2 + y^2)$.

at (a,b)

$$P_2(x,y) = f(a,b) + \vec{\nabla} f(a,b) \cdot \begin{bmatrix} x-a \\ y-b \end{bmatrix} + \frac{1}{2}[x-a, y-b] \, Hf(a,b) \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

a) $f(0,0) = e^0 \ln(1+0) = 1 \cdot \ln(1) = 0$.

$$\vec{\nabla} f(0,0) = \left[e^x \ln(1+y) \Big|_{(0,0)}, \ \frac{e^x}{1+y} \Big|_{(0,0)} \right]$$

$$= [0, 1].$$

$$Hf(0,0) = \begin{bmatrix} e^x \ln(1+y) \Big|_{(0,0)} & \frac{e^x}{1+y} \Big|_{(0,0)} \\ \frac{e^x}{1+y} \Big|_{(0,0)} & \frac{-e^x}{(1+y)^2} \Big|_{(0,0)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

So $P_2(x,y) = 0 + [0,1] \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2}[x, y] \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$= y + \frac{1}{2}[x, y] \begin{bmatrix} y \\ x-y \end{bmatrix}$$

$$= y + \frac{1}{2}xy + \frac{1}{2}y(x-y) = y + xy - \frac{1}{2}y^2$$

b) $f(0,0) = \sin(0^2+0^2) = 0.$

$$\vec{\nabla} f(0,0) = \left[2x\cos(x^2+y^2)\Big|_{(0,0)} \quad 2y\cos(x^2+y^2)\Big|_{(0,0)} \right] = [0,0].$$

$$Hf(0,0) = \begin{bmatrix} 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2)\Big|_{(0,0)} & 4xy\sin(x^2+y^2)\Big|_{(0,0)} \\ -4xy\sin(x^2+y^2)\Big|_{(0,0)} & 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2)\Big|_{(0,0)} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

So $P_2(x,y) = 0 + \cancelto{0}{[0,0]\begin{bmatrix} x \\ y \end{bmatrix}} + \frac{1}{2}[x \; y]\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$= \frac{1}{2}[x \; y]\begin{bmatrix} 2x \\ 2y \end{bmatrix} = x^2+y^2$

4. Find the 2nd order Taylor polynomial of $f(x, y) = 4xy - x^4 - y^4$ near $(0, 0)$ and also find and classify all of its critical points using the Second Derivative Test.

$f(0,0) = 0$. $\vec{\nabla} f(0,0) = \left[4y-4x^3\big|_{(0,0)}, 4x-4y^3\big|_{(0,0)}\right] = [0, 0]$.

$$Hf(0,0) = \begin{bmatrix} -12x^2\big|_{(0,0)} & 4 \\ 4 & -12y^2\big|_{(0,0)} \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}.$$

So $P_2(x,y) = 0 + [0,0]\begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2}[x, y]\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$= \frac{1}{2}[x, y]\begin{bmatrix} 4y \\ 4x \end{bmatrix} = 2xy + 2yx = 4xy$.

Critical Pts: $\begin{cases} 4y - 4x^3 = 0 & (1) \\ 4x - 4y^3 = 0 & (2) \end{cases}$ $(1) \Rightarrow y = x^3$

Sub into (2): $4x - 4(x^3)^3 = 0$

$\Rightarrow 4x - 4x^9 = 0$

$\Rightarrow 1 = x^8$ or $x = 0 \Rightarrow y = 0$

$1 = x^8 \Rightarrow x = 1$, or $x = -1$

$\Rightarrow y = 1$ $\quad \Rightarrow y = -1$.

So $(0,0)$, $(1,1)$, $(-1,-1)$ are critical points.

$Hf(0,0) = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$ $f_{xx} = 0$, $\det = -16 < 0 \Rightarrow (0,0)$ is a saddle pt.

$Hf(1,1) = \begin{bmatrix} -12 & 4 \\ 4 & -12 \end{bmatrix}$, $f_{xx} = -12 < 0$, $\det = (-12)^2 - 4^2 = 128 > 0$

$\Rightarrow (1,1)$ is a local max.

$Hf(-1,-1) = \begin{bmatrix} -12 & 4 \\ 4 & -12 \end{bmatrix}$ $\Rightarrow (-1,-1)$ is also a local max.